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π- CLOSED SETS AND ALMOST NORMALITY OF THE NIEMYTZKI PLANE TOPOLOGY

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Abstract

The Niemytzki plane topology is a famous example of a Tychonoff space, which is quasi-normal but not normal. In this paper, we show that it is almost normal but not semi-normal. Also, we give some short notes on a class of sets in topological spaces called π-closed sets, which introduced by Zaitsev in 1968. This class of sets is strictly placed between closed domain sets and closed sets. Practically, some basic properties of π-closed sets are presented.

1. Introduction

Throughout this paper, a space X always means a topological space on which no separation axioms are assumed unless explicitly stated. We will denote an ordered pair by $\langle x, y\rangle$ and the set of real numbers by $\mathbb{R}$. For a subset A of a space X, $\overline{A}$ and $\operatorname{int}(A)$ denote to the closure and the interior of A in X, respectively. A subset A of a topological space X is

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called a *closed domain*, if $A = \overline{\text{int}(A)}$, see [10]. A subset A of X is called an *open domain*, if $A = \text{int}(\overline{A})$, see [10]. A subset A of a space X is called a π*-closed*, if it is a finite intersection of closed domain subsets. A subset A of a space X is called a π*-open,* if it is a finite union of open domain subsets, see [21]. Two sets A and B of a topological space X are said to be *separated*, if there exist disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see [2, 3]. A topological space X is called a *mildly normal*, if any two disjoint closed domain subsets A and B of X can be separated, see [18]. A space X is called an *almost normal*, if any two disjoint closed subsets A and B of X, one of which is closed domain, can be separated, see [17]. A space X is called a *quasi-normal*, if any disjoint π-closed subsets A and B of X can be separated, see [21]. A space X is said to be a *semi-normal*, if for any closed set A and every open set B with $A \subseteq B$, there exists an open set U such that $A \subseteq U \subseteq \text{int}(\overline{U}) \subseteq B$, see [17]. Zaitsev in [21], proved that the Niemytzki plane topology is quasi-normal. Hence, it is mildly normal but not normal. Kalantan in [9], presented the following open problems:

Problem 1. Is the Niemytzki plane topology almost normal space?

Problem 2. Is the Sorgenfrey line square topology almost normal?

In this paper, we give some basic properties of π-closed sets. Also, we show that the Niemytzki plane topology and the Sorgenfrey line square topology are almost normal and not semi-normal spaces.

2. Short Notes on π-Closed Sets

In this section, we give some basic properties of π-closed sets in a space X. We begin to present the following proposition, which can be proved easily:

Proposition 2.1. *Let* $(X, \mathcal{T})$ *be a topological space and* $M \subseteq X$.

(1) *If M is an open domain in X and* $A \subseteq M$, *then A is* π*-open set in M, if and only if A is* π*-open set in X.*

(2) *Let M be a closed domain in X and* $A \subseteq M$. *If A is* π*-closed set in M, then A is* π*-closed set in X.*

The converse of the Proposition 2.1 part (2) need not be true in general. Counterexamples can be found in the finite topological spaces. It is obvious that, if M is a closed-and-open (clopen) subset of X and $G \subseteq X$ is closed domain, then $M \cap G$ is π-closed set in M. Now, we prove the following result:

Proposition 2.2. *Let* $(X, \mathcal{T})$ *be a topological space,* $M \subseteq X$ *and* $G \subseteq X$. *If M is a clopen subset of X and G is* π*-closed set in X, then* $M \cap G$ *is* π*-closed set in M.*

Proof. Suppose that M is a clopen subset of X and G is a π-closed set in X. Then $G = \bigcap_{i=1}^{n} D_i$, where D_i is closed domain in X, for all $i \in \{1, 2, 3, \ldots, n\}$. So $M \cap D_i$ is closed domain in M, for all $i \in \{1, 2, 3, \ldots, n\}$. Since the finite intersection of π-closed sets is π-closed, thus $\bigcap_{i=1}^{n} (M \cap D_i)$ is π-closed set in M. But $\bigcap_{i=1}^{n} (M \cap D_i) = M \cap (\bigcap_{i=1}^{n} D_i) = M \cap G$. Hence, $M \cap G$ is π-closed set in M. □

Observe that if M is π-closed in X and $A \subseteq X$, then:

(1) If A is π-closed set in X, then $M \cap A$ is not necessary to be π-closed set in M.

(2) If $A \subseteq M$ and A is a π-closed set in M, then it is not necessary that A is π-closed set in X.

Counterexamples can be found in finite spaces.

Now, we will present the following properties without proof:

Proposition 2.3. (1) *The image of a* π*-closed subset under a closed-and-open, one-to-one, continuous function is* π*-closed subset.*

(2) *The inverse image of a* π*-closed* (π*-open*) *subset under an open, continuous function is* π*-closed* (π*-open*).

We can observe that the image of a π-closed set under a closed function is not necessary to be π-closed, the counterexample is shown below:

Example 2.4. Consider $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{CF})$ such that $f(x) = 1$, for each $x \in \mathbb{R}$. f is a function from $\mathbb{R}$ with its usual topology to $\mathbb{R}$ with its co-finite topology. We observe that f is closed function. In fact, if F be any closed set in $(\mathbb{R}, \mathcal{U})$, we have $f(F) = \{1\}$, which is closed in $(\mathbb{R}, \mathcal{CF})$. Now, if A be any π-closed set in $(\mathbb{R}, \mathcal{U})$, then $f(A) = \{1\}$, which is not π-closed set in $(\mathbb{R}, \mathcal{CF})$, since the only π-closed sets in $(\mathbb{R}, \mathcal{CF})$ are $\mathbb{R}$ and $\emptyset$.

The following example shows that the image of a π-closed (π-open) subset under an open function is not necessary to be π-closed (π-open).

Example 2.5. Consider $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ such that $f(x) = \sqrt{2}$, for each $x \in \mathbb{R}$. $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is the particular point topology, where $\mathcal{T}_{\sqrt{2}} = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : \sqrt{2} \in U\}$. f is an open function. In fact, if U be any non-empty open set in $(\mathbb{R}, \mathcal{U})$, then $f(U) = \{\sqrt{2}\}$, which is an open set in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$. In $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$, the closure of any non-empty open set is equal to $\mathbb{R}$ and the interior of any proper closed set is equal to $\emptyset$. So, there are only two open domain (resp., closed domain) subsets in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$, which are $\mathbb{R}$ and $\emptyset$. Hence, there are only two π-open (resp., π-closed) subsets in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$, which are $\mathbb{R}$ and $\emptyset$. Now, if A be any non-empty π-closed subset in $(\mathbb{R}, \mathcal{U})$, we have $f(A) = \{\sqrt{2}\}$, which is not π-closed in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$. If B be any non-empty π-open subset in $(\mathbb{R}, \mathcal{U})$, we have $f(B) = \{\sqrt{2}\}$, which is not π-open subset in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$.

The following example shows that the inverse image of a π-closed (π-open) subset under a closed function is not necessary to be π-closed (π-open).

Example 2.6. Consider $f : (\mathbb{R}, \mathcal{L}) \to (\mathbb{R}, \mathcal{U})$ such that $f(x) = x$, for each $x \in \mathbb{R}$. f is closed function because $\mathcal{L} \subseteq \mathcal{U}$, where $\mathcal{L}$ is the left ray topology on $\mathbb{R}$, see [19]. If $A = [0, 1]$, then A is π-closed subset in $(\mathbb{R}, \mathcal{U})$. But $f^{-1}(A) = [0, 1]$ is not π-closed in $(\mathbb{R}, \mathcal{L})$, because it is not closed. If $B = (0, 1)$, then B is π-open in $(\mathbb{R}, \mathcal{U})$. But $f^{-1}(B) = (0, 1)$ is not π-open in $(\mathbb{R}, \mathcal{L})$, because it is not open.

Now, we will prove that a finite product of π-closed sets is π-closed.

Proposition 2.7. *Let X_i be a topological space, for all $i \in \{1, 2, 3, \dots, n\}$, $n \in \mathbb{N}$. If A_i is π-closed subset of X_i for all $i \in \{1, 2, 3, \dots, n\}$, then $\prod_{i=1}^{n} A_i$ is π-closed subset of $\prod_{i=1}^{n} X_i$.*

Proof. Let A_i be a π-closed subset of X_i, for all $i \in \{1, 2, 3, \dots, n\}$. Write $A_i = \cap_{j=1}^{m_i} F_{ij}$, where F_{ij} is closed domain in X_i, for each $j \in \{1, 2, 3, \dots, m_i\}$. Let $k = \max\{m_1, m_2, m_3, \dots, m_n\}$. Then, we can write $A_i = \cap_{i=1}^{k} F_{ij}$, where $F_{ij} = X_i$, if $j > m_i$. Thus, $\prod_{i=1}^{n} A_i = \prod_{i=1}^{n} (\cap_{j=1}^{k} F_{ij})$ $= \cap_{j=1}^{k} \prod_{i=1}^{n} F_{ij}$. Since F_{ij} is closed domain in X_i, for each $i \in \{1, 2, 3, \dots, n\}$ and the finite product of closed domain sets is closed domain, see [3], then $\prod_{i=1}^{n} F_{ij}$ is closed domain in $\prod_{i=1}^{n} X_i$. Hence, $\prod_{i=1}^{n} A_i$ is π-closed in $\prod_{i=1}^{n} X_i$. □

3. Almost Normality of the Niemytzki Plane and the Sorgenfrey Line Square Topological Spaces

In this section, we recall the definitions of two famous topological spaces, which are the Niemytzki plane topology and the Sorgenfrey line square topology, see [19]. Some properties of these spaces are presented.

(1) If $\mathbb{P} = \{\langle x, y\rangle : x, y \in \mathbb{R}, y > 0\}$ is the open upper half-plane with the usual Euclidean topology. If $\mathbb{L}$ is the x-axis, we generate a topology $\mathcal{T}$ on $X = \mathbb{P} \cup \mathbb{L}$ by the following neighborhood system:

The basic open neighborhood of $\langle x, y\rangle \in \mathbb{P}$ is an open disc D in $\mathbb{P}$. The basic open neighborhood of $\langle x, 0\rangle \in \mathbb{L}$ is of the form $\{\langle x, 0\rangle\} \cup D$, where D is an open disc in $\mathbb{P}$, which is tangent to $\mathbb{L}$ at the point $\langle x, 0\rangle$. This topology is called the *Niemytzki plane* topology or the *Moore plane*.

(2) Consider the set of real numbers $\mathbb{R}$ and let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}, a < b\}$, then $\mathcal{B}$ is a base for a unique topology on $\mathbb{R}$. This topology is called the *Sorgenfrey line* topology and denoted by $\mathcal{S}$.

Proposition 3.1. *In the Niemytzki plane topology, the subset* $\mathbb{L} = \{\langle x, 0\rangle : x \in \mathbb{R}\}$ *is* π*-closed and any singleton* $\{\langle x, y\rangle\}$ *is* π*-closed.*

Proof. For each $n \in \mathbb{N}$, let $A_n = \{\langle x, y\rangle : x \in \mathbb{R}, \frac{1}{4n-2} < y < \frac{1}{4n-3}\}$ and let $B_n = \{\langle x, y\rangle : x \in \mathbb{R}, \frac{1}{4n} < y < \frac{1}{4n-1}\}$. Then A_n and B_n are disjoint open sets in X, for each $n \in \mathbb{N}$. Furthermore, $\overline{A_n}$ and $\overline{B_n}$ are also disjoint. Let $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$. Then A and B are open sets in X, where $A \cap B = \emptyset$. Now $\overline{A} = (\bigcup_{n\in\mathbb{N}} \overline{A_n}) \cup \mathbb{L}$ and $\overline{B} = (\bigcup_{n\in\mathbb{N}} \overline{B_n}) \cup \mathbb{L}$, where $\overline{A_n} = \{\langle x, y\rangle : x \in \mathbb{R}, \frac{1}{4n-2} \leq y \leq \frac{1}{4n-3}\}$ and $\overline{B_n} = \{\langle x, y\rangle : x \in \mathbb{R}, \frac{1}{4n} \leq y \leq \frac{1}{4n-1}\}$. Thus, $\overline{A}$ and $\overline{B}$ are closed domains in X and $\mathbb{L} = \overline{A} \cap \overline{B}$. Therefore, $\mathbb{L}$ is π-closed subset of X.

Now, we need to prove that any singleton $\{\langle x, y\rangle\}$ is π-closed.

Pick any $\langle x, 0\rangle \in X$. Let $D \cup \{\langle x, 0\rangle\}$ be any basic open set of $\langle x, 0\rangle$ in X. Then, we have $\overline{D}$ is closed disc in $\mathbb{P}$ with radius $r > 0$ and tangent $\mathbb{L}$ at $\langle x, 0\rangle$. Thus $\overline{D} \cap \mathbb{L} = \{\langle x, 0\rangle\}$. Since $\overline{D}$ is closed domain in X (hence π-closed), $\mathbb{L}$ is also π-closed and the intersection of two π-closed sets is also π-closed, then $\overline{D} \cap \mathbb{L}$ is π-closed. Hence $\{\langle x, 0\rangle\}$ is π-closed.

Now, let $\langle x, y\rangle$, $y > 0$ be any point in $\mathbb{P}$. Then $D = B_1(\langle x + y, y\rangle; y)$ is an open disc in $\mathbb{P}$ with radius y and centered at the point $\langle x + y, y\rangle$. Also, $E = B_2(\langle x - y, y\rangle; y)$ is an open disc in $\mathbb{P}$ with radius y and centered at the point $\langle x - y, y\rangle$. We observe that D and E are disjoint open sets. But $\overline{D} \cap \overline{E} = \{\langle x, y\rangle\}$, where $\overline{D}$ and $\overline{E}$ are closed domain sets in X. Hence $\{\langle x, y\rangle\}$ is π-closed. Therefore, any singleton $\{\langle x, y\rangle\}$ is π-closed set in the Niemytzki plane topology. □

Proposition 3.2. *In the Sorgenfrey line square topology* $(\mathbb{R}^2, \mathcal{S}^2)$, *the set* $L = \{\langle x, -x\rangle; x \in \mathbb{R}\}$ *is* π*-closed and any singleton of the form* $\{\langle x, -x\rangle\}$, $x \in \mathbb{R}$ *is* π*-closed.*

Proof. For each $n \in \mathbb{N}$, let $L_n = \{\langle x, y\rangle : y = -x + \frac{1}{n}\}$. Now, for each $n \in \mathbb{N}$, let $A_n = \{\langle x, y\rangle : -x + \frac{1}{4n-2} < y < -x + \frac{1}{4n-3}\}$ and $B_n = \{\langle x, y\rangle : -x + \frac{1}{4n} < y < -x + \frac{1}{4n-1}\}$. Then A_n and B_n are open sets in $(\mathbb{R}^2, \mathcal{S}^2)$ and $\overline{A_n} \cap \overline{B_n} = \emptyset$, for each $n \in \mathbb{N}$. Now, put $A = \bigcup_{n\in\mathbb{N}} A_n$ and $B = \bigcup_{n\in\mathbb{N}} B_n$. Then A and B are open sets in $(\mathbb{R}^2, \mathcal{S}^2)$ and $A \cap B = \emptyset$. Since $\{A_n : n \in \mathbb{N}\}$ is a family of pairwise disjoint open sets, then by locally finiteness, we have $\overline{A} = \overline{\bigcup_{n\in\mathbb{N}} A_n} = (\bigcup_{n\in\mathbb{N}} \overline{A_n}) \cup L$. Similarly, $\overline{B} = \overline{\bigcup_{n\in\mathbb{N}} B_n} = (\bigcup_{n\in\mathbb{N}} \overline{B_n}) \cup L$. Since for each $n \in \mathbb{N}$, $\overline{A_n} \cap \overline{B_n} = \emptyset$, then $(\bigcup_{n\in\mathbb{N}} \overline{A_n}) \cap (\bigcup_{n\in\mathbb{N}} \overline{B_n}) = \emptyset$. Thus $\overline{A} \cap \overline{B} = L$, where $\overline{A}$ and $\overline{B}$ are closed domain sets in $(\mathbb{R}^2, \mathcal{S}^2)$. Hence L is π-closed.

Now, we prove that any singleton $\{\langle x, -x\rangle\}$, $x \in \mathbb{R}$ is π-closed.

For each $x \in \mathbb{R}$, we have $\{\langle x, -x\rangle\} = ([x, x+1) \times [-x, -x+1)) \cap L$. Since the product $[x, x+1) \times [-x, -x+1)$ is closed domain (hence π-closed) and L is π-closed, then we have $([x, x+1) \times [-x, -x+1)) \cap L$ is π-closed. Hence $\{\langle x, -x\rangle\}$ is π-closed. □

It is easy to prove the following property:

Proposition 3.3. *In the Sorgenfrey line topology* $(\mathbb{R}, \mathcal{S})$, *any singleton* $\{x\}$ *is* π*-closed and any singleton* $\{\langle x, y\rangle\}$ *is* π*-closed subset in* $(\mathbb{R}^2, \mathcal{S}^2)$.

Now, we prove the following result:

Proposition 3.4. *The Niemytzki plane topology is almost normal space.*

Proof. Let A and B be any disjoint closed sets in X such that A is closed domain. We observe that, if B is π-closed, then A and B are disjoint π-closed subsets of X. Since the Niemytzki plane topology is quasi-normal, then A and B can be separated. Now, suppose that B is not π-closed. Since A is closed domain, then $A = \overline{\operatorname{int}(A)}$. Thus A can not be in $\mathbb{L}$, because for each $A \subseteq \mathbb{L}$, we have $\overline{\operatorname{int}(A)} = \emptyset \neq A$. So, each closed set A in X with $A \subseteq L$ is not closed domain. Therefore $A \not\subseteq \mathbb{L}$. Then either $A \subset \mathbb{P}$ or $(A \cap \mathbb{L} \neq \emptyset$ and $A \cap \mathbb{P} \neq \emptyset)$. For each case about A, there are three subcases about B, which are $B \subset \mathbb{P}$ or $B \subset \mathbb{L}$ or $(B \cap \mathbb{P} \neq \emptyset$ and $B \cap \mathbb{L} \neq \emptyset)$. Now, we show that A and B can be separated for each case.

Case 1. Let $A \subset \mathbb{P}$.

Subcase 1a. Let $B \subset \mathbb{P}$.

Since $A = A \cap \mathbb{P}$ and $B = B \cap \mathbb{P}$, then A and B are disjoint closed sets in $\mathbb{P}$ with its usual Euclidean topology. Thus, there exist disjoint open sets U and V in $\mathbb{P}$ such that $A \subseteq U$ and $B \subseteq V$. Since $\mathbb{P}$ is an open subspace of X, then U and V are also open sets in X. Hence A and B can be separated.

Subcase 1b. Let $B \subset \mathbb{L}$.

Since $A \subset \mathbb{P}$, then $A \cap \mathbb{L} = \emptyset$. So A and $\mathbb{L}$ are disjoint π-closed sets in X. By quasi-normality of X, there exist disjoint open sets U and V of X such that $A \subseteq U$ and $\mathbb{L} \subseteq V$. Since $B \subset \mathbb{L}$, then we have $A \subseteq U$ and $B \subseteq V$. Therefore A and B can be separated.

Subcase 1c. Let $B \cap \mathbb{L} \neq \emptyset$ and $B \cap \mathbb{P} \neq \emptyset$.

Since $A \subset \mathbb{P}$ and $A \cap B = \emptyset$, then $A \cap (B \cap \mathbb{P}) = \emptyset$. A and $B \cap \mathbb{P}$ are disjoint closed sets in $\mathbb{P}$ with its usual Euclidean topology. By normality of $\mathbb{P}$, there exist disjoint open sets U_1 and V_1 in $\mathbb{P}$ and hence in X such that

$$A \subseteq U_1 \text{ and } B \cap \mathbb{P} \subseteq V_1. \tag{1}$$

Also, $A \cap (B \cap \mathbb{L}) = \emptyset$, $B \cap \mathbb{L} \subset \mathbb{L}$ is closed set in X. By Subcase 1b, there exist disjoint open sets U_2 and V_2 in X such that

$$A \subseteq U_2 \text{ and } B \cap \mathbb{L} \subseteq V_2. \tag{2}$$

From (1) and (2), we have $A \subseteq U_1 \cap U_2$ and $B \subseteq V_1 \cup V_2$. Put $U = U_1 \cap U_2$ and $V = V_1 \cup V_2$. Then U and V are disjoint open sets in X such that $A \subseteq U$ and $B \subseteq V$. Hence A and B can be separated.

Case 2. Let $A \cap \mathbb{L} \neq \emptyset$ and $A \cap \mathbb{P} \neq \emptyset$.

Subcase 2a. Let $B \subset \mathbb{P}$.

Since $A \cap B = \emptyset$, then $(A \cap \mathbb{P}) \cap B = \emptyset$. Thus, $A \cap \mathbb{P}$ and B are disjoint closed sets in $\mathbb{P}$ with its usual Euclidean topology. By normality of $\mathbb{P}$, there exist disjoint open sets U_1 and V_1 in $\mathbb{P}$ and hence in X such that

$$A \cap \mathbb{P} \subseteq U_1 \text{ and } B \subseteq V_1. \tag{3}$$

Since A is closed domain and $\mathbb{P}$ is an open dense subset of X, then we have $A = \overline{A \cap \mathbb{P}} = \overline{\text{int}(A) \cap \mathbb{P}}$. Now by (3), we have $\overline{A \cap \mathbb{P}} \subseteq \overline{U_1}$ and thus $A \subseteq \overline{U_1}$. Also, by open density and normality of $\mathbb{P}$, it is easy to show that $\overline{U_1} \cap \overline{V_1} = \emptyset$. Thus $A \cap \overline{V_1} = \emptyset$. Therefore, A and $\overline{V_1}$ are disjoint closed domain subsets of X. By mild-normality of X, there exist disjoint open subsets U_2 and V_2 of X such that $A \subseteq U_2$ and $\overline{V_1} \subseteq V_2$. Now, put $U = U_2$ and $V = V_1 \cap V_2$. Then U and V are disjoint open subsets of X such that $A \subseteq U$ and $B \subseteq V$. Hence A and B can be separated.

Subcase 2b. Let $B \subset \mathbb{L}$.

Since $A \cap \mathbb{P} \neq \emptyset \neq A \cap \mathbb{L}$, A is closed domain and $A \cap B = \emptyset$, then $\overline{A \cap \mathbb{P}} = A$ and $B \cap (\overline{A \cap \mathbb{P}}) = \emptyset$. Then for each $\langle x, 0\rangle \in B$, we have $\langle x, 0\rangle \notin \overline{A \cap \mathbb{P}}$. Therefore, there exists a basic open neighborhood $D_x(r_x)$ of $\langle x, 0\rangle$ with radius $r_x > 0$ and tangent $\mathbb{L}$ at $\langle x, 0\rangle$ such that $D_x(r_x) \cap (A \cap \mathbb{P}) = \emptyset$. This implies that $D_x(r_x) \cap A = \emptyset$. Since $D_x(r_x)$ and $\overline{D_x(r_x)}$ have at most one element in $\mathbb{L}$, then $\overline{D_x(r_x)} \cap A = \emptyset$, for each $\langle x, 0\rangle \in B$. Now, $B \subseteq \bigcup_{\langle x, 0\rangle \in B} D_x(r_x)$. Put $V = \bigcup_{\langle x, 0\rangle \in B} D_x(r_x)$. Then, V is an open set of X such that $B \subseteq V$ and $V \cap A = \emptyset$. Since $A = \overline{\operatorname{int}(A)}$, then $V \cap \operatorname{int}(A) = \emptyset$. Hence $\overline{V} \cap \operatorname{int}(A) = \emptyset$. So, there exists an open set V of X such that

$$B \subseteq V, \quad V \cap A = \emptyset, \quad \text{and } \overline{V} \cap \operatorname{int}(A) = \emptyset. \tag{4}$$

Since every closed set in X is a G_δ-set, see [3], then B is G_δ-set of X. Since X is first countable and B is G_δ-set of X, then there exists a decreasing sequence $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots \supseteq U_n \supseteq \ldots$ of open sets of X such that $B = \bigcap_{n\in\mathbb{N}} U_n$. Thus $B \subseteq U_n$, for each $n \in \mathbb{N}$. Now, $B \subseteq U_n \cap V$. Therefore $U_n \cap V \neq \emptyset$, for each $n \in \mathbb{N}$. Put $V_n = U_n \cap V$. Then $\{V_n : n \in \mathbb{N}\}$ is decreasing sequence of open sets such that $B = \bigcap_{n\in\mathbb{N}} V_n$.

Claim. There exists an $m \in \mathbb{N}$ such that $\overline{V_m} \cap A = \emptyset$.

Suppose that $\overline{V_n} \cap A \neq \emptyset$, for each $n \in \mathbb{N}$. Then, there exists an element $y \in X$ such that $y \in A$ and $y \in \overline{V_n}$, for each $n \in \mathbb{N}$. Since $V_n \subseteq V$, then by (4), we have $y \notin \operatorname{int}(A)$ and $y \notin V_n$, for each n. Now, since $y \in \overline{V_n}$, for each $n \in \mathbb{N}$, then $y \in \mathbb{L}$ and for any basic open neighborhood D_y of y, we have

$$D_y \cap V_n \neq \emptyset, \text{ for each } n \in \mathbb{N}. \tag{5}$$

Clearly, that for each $n \in \mathbb{N}$, $D_y \cap V_n \cap \mathbb{L} = \emptyset$.

If $D_y \cap V_k \cap \mathbb{L} \neq \emptyset$, for some $k \in \mathbb{N}$, then there exists an element z in $\mathbb{L}$ such that $z \in D_y$ and $z \in V_k$. Since D_y has at most one element in $\mathbb{L}$ and $y \in \mathbb{L}$, then $y = z$. Thus $y \in V_k$. Since $y \in A$, then $V_k \cap A \neq \emptyset$, which is a contradiction, by (5). Therefore, for each $n \in \mathbb{N}$, $D_y \cap V_n \cap \mathbb{L} = \emptyset$. Thus $D_y \cap V_n \subset \mathbb{P}$, for each n. Now, since $D_y \cap V_n \neq \emptyset$, for each n, then there exists an element $z \in D_y$ and $z \in V_n$, for each n. Thus $z \in \bigcap_{n\in\mathbb{N}} V_n = B$. Hence $z \in B \subseteq \mathbb{L}$. But D_y has at most one element in $\mathbb{L}$. Since $z \in D_y$, then $z = y$. Hence $y \in B$. But $y \in A$ and therefore $A \cap B \neq \emptyset$, which is a contradiction as $A \cap B = \emptyset$. Thus if $\overline{V_n} \cap A \neq \emptyset$, for each n, we have a contradiction. Therefore, there exists an $m \in \mathbb{N}$ such that $\overline{V_m} \cap A = \emptyset$.

Now, A and $\overline{V_m}$ are disjoint closed domain sets of X. By mild normality of X, there exist disjoint open sets G and H of X such that $A \subseteq G$ and $\overline{V_m} \subseteq H$. Since $B \subseteq \overline{V_m}$, then we have two disjoint open sets G and H of X such that $A \subseteq G$ and $B \subseteq H$. Hence A and B can be separated.

Subcase 2c. Let $B \cap \mathbb{L} \neq \emptyset \neq B \cap \mathbb{P}$.

Since $A \cap B = \emptyset$, then $(A \cap \mathbb{P}) \cap (B \cap \mathbb{P}) = \emptyset$. Thus, $A \cap \mathbb{P}$ and $B \cap \mathbb{P}$ are disjoint closed sets in $\mathbb{P}$ with its usual Euclidean topology. So, by normality of $\mathbb{P}$, there exist open sets U_1 and V_1 in $\mathbb{P}$ and hence in X such that

$$A \cap \mathbb{P} \subseteq U_1, \quad B \cap \mathbb{P} \subseteq V_1, \quad \text{and } \overline{U_1} \cap \overline{V_1} = \emptyset. \tag{6}$$

Since A is closed domain and $\mathbb{P}$ is open dense subset of X, then $\overline{A \cap \mathbb{P}} = A$. Thus $A \subseteq \overline{U_1}$. By (6), we have $A \cap \overline{V_1} = \emptyset$. Thus A and $\overline{V_1}$ are disjoint closed domain of X. By mild normality of X, there exist disjoint open sets U_2 and V_2 of X such that $A \subseteq U_2$ and $\overline{V_1} \subseteq V_2$. Thus, we have

$$A \subseteq U_2, \quad B \cap \mathbb{P} \subseteq V_2, \quad \text{and } U_2 \cap V_2 = \emptyset. \tag{7}$$

Since $A \cap (B \cap \mathbb{L}) = \emptyset$ and $B \cap \mathbb{L} \subset \mathbb{L}$ is closed subset of X, then by Subcase 2b, there exist open sets U_3 and V_3 of X such that

$$A \subseteq U_3, \quad B \cap \mathbb{L} \subseteq V_3, \quad \text{and } U_3 \cap V_3 = \emptyset. \tag{8}$$

By (7) and (8), put $G = U_2 \cap U_3$ and $H = V_2 \cup V_3$. Then G and H are disjoint open sets of X such that $A \subseteq G$ and $B \subseteq H$. Hence A and B can be separated.

For all cases, A and B can be separated by two disjoint open subsets of X. Hence X is almost normal space. □

Since every almost normal, semi-normal space is normal, see [17], then we have the following corollary:

Corollary 3.5. *The Niemytzki plane topology is not semi-normal space.*

Proposition 3.6. *The Sorgenfrey line square topology is almost normal and not semi-normal space.*

Proof. We can prove this proposition as the same argument as that of the Proposition 3.4. □

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